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An approach to turbulent flame theory

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Is it possible to express the problem of calculating turbulent flame speeds as an eigenvalue problem that is analogous to the laminar flame speed problem? It is argued for grid turbulence that the answer is affirmative, and some benefits of pursuing such a calculation are exploited for the limiting case of a first-order reaction with vanishingly small heat release. The streamwise turbulent transport of reactant occupies a central role in the analysis. The equation governing the ensemble average of this quantity assumes different simplified forms in the limits of small-scale and large-scale turbulence. The criterion which is obtained for separating the small-scale and large-scale régimes differs from that of Damköhler and also from that of Kovasznay and Klimov. In the small-scale régime, turbulence produces a spatially varying diffusivity, the form of which can be ascertained only through an investigation of non-linear equations describing the statistical dynamics of production and decay of the velocity–concentration correlation. In the large-scale régime, which is of greater practical importance, the ensemble average of the streamwise turbulent reactant flux satisfies a linear ordinary differential equation whose solution for the growth and decay of the flux contains effects resembling wrinkling of the laminar flame, increasing of the effective diffusivity and augmentation of the effective reaction rate. An exact solution to the linear eigenvalue problem which arises in the large-scale limit reveals that turbulence enhances mean reactant consumption in the upstream portion of the flame and retards reactant consumption downstream. Formulas are given for the increase in flame speed and the increase in flame thickness that are produced by turbulence in the large-scale limit. Since the equations are relatively tractable in the large-scale limit, it is suggested that further study of these equations may yield improved descriptions of realistic turbulent flames.

1. Introduction

Turbulent reacting flows are difficult to analyze. While questions remain concerning the dynamics of turbulent decay downstream from a grid, in comparison the dynamics of turbulent flames are a complete enigma. The first true advance in our understanding of turbulent flame propagation can be attributed to Damköhler (1940) who reasoned that turbulence should increase transport rates within a flame for turbulent scales sufficiently small compared with the laminar flame thickness and wrinkle the laminar flame without affecting its internal dynamics for sufficiently large turbulent scales. Based only on the observation that in engineering applications the scale of energy-containing eddies usually exceeds the thickness of a laminar flame, many flame theoreticians attempted to calculate turbulent flame speeds by analyzing the motion of a wrinkled laminar flame in a turbulent field. The most thorough of these analyses were performed by Tucker (1956) and by Richardson (1956).

Tucker's calculation, which can be viewed as a generalization of Batchelor's (1953, pp. 68–75) calculation of the change in a turbulent field produced by a sudden contraction, exhibits the disturbing result that the turbulent flame speed approaches infinity as the heat released by the flame approaches zero. Perhaps this anomaly is a consequence of the linearization in which the angle between the local normal to the laminar flame and the direction of propagation of the turbulent flame is presumed small, but why the linearization should fail only for small heat release is unclear. Richardson's results share the small-angle assumption while exhibiting no such anomaly, but they are unrealistic in that for mathematical convenience the turbulent velocity field is approximated as strictly one-dimensional. Both analyses, and indeed all published wrinkled laminar flame theories of turbulent flames, treat the local laminar flame speed as a constant, a practice which may be hazardous since Landau (1944), Markstein (1951) and others have shown that under this condition plane laminar flames are unstable to disturbances of all wavelengths in the approach flow. The stability analyses demonstrate that when a reasonable dependence of flame speed on flame curvature is introduced, laminar flames are stable only to disturbances of wavelength smaller than a critical value. If the plane laminar flame is unstable to some of the components comprising the incident turbulent field, then what significance emerges from a calculation of the response of the laminar flame to such a turbulence? The future of wrinkled laminar flame theories for turbulent flames will become promising only after improvements are achieved in understanding the dynamical behaviour of laminar flames in non-uniform velocity fields.

An important step toward improving this understanding has recently been taken by Klimov (1963) who studied laminar flame structure and laminar flame speeds in flow regions of uniform shear. Klimov demonstrated that non-uniform velocity fields modify both flame structure and flame speed. He discovered that the character of the modifications depends on the value of a quantity γ which is defined as the ratio of a representative velocity gradient† in the turbulent field to the reciprocal of a representative residence time in the flame. For small values of this parameter, changes in flame speed and flame structure are small but non-zero; for large values phenomena such as negative flame speeds and flame extinction occur.

At the present time it would be interesting to investigate the relationship of Klimov's results to the stability analyses of Landau and others and to proceed from there to a proper theory of the wrinkled laminar flame model of turbulent

† The gradient in the direction normal to the flame of the component of velocity normal to the flame, or, equivalently, the time rate of change of the logarithm of the area of an element of the flame surface.

flames. This probably can be accomplished only for small values of γ since Klimov's analysis tends to indicate that turbulent flames with large values of γ will be too complex to be analyzed in terms of laminar flame dynamics. Instead of pursuing the development of wrinkled flame theory for small values of γ, we initiate in this paper an alternative approach to turbulent flame theory which in principle is not restricted *a priori* to small values of γ.

In place of a detailed description of the dynamics of flamelets within the turbulent flame brush, we content ourselves with a partial statistical description of the turbulent flame interior, in the hope that the description can be made complete enough to reveal certain gross flame characteristics. The gross characteristic of primary practical interest is the turbulent flame speed, and the nearly single-minded objective of the present study is to determine the turbulent flame speed. In this respect, the present study differs from other studies in the statistical theory of turbulence, many of which appear to be directed toward discovering anything that can be discovered about turbulence. There is a resemblance between the present study and, for example, that of Corrsin (1952) whose main objective was to determine the turbulent heat transfer coefficient.† But there is no resemblance between the present study and those focusing on, say, two-point correlations and their spectral resolutions, because from the viewpoint of the present analysis these quantities are not of direct relevance to the turbulent flame speed.

After satisfying ourselves, with certain reservations, that a turbulent flame speed can be defined in a manner completely analogous to the way in which laminar flame speeds are defined, we specialize our analysis to the case of a first-order reaction with negligible heat release. The purpose of the specialization is to produce simpler equations that reveal more easily some fundamental aspects of the approach. The simplified equation for the average streamwise turbulent reactant transport possesses two limiting cases—the limits of small and large turbulence scales. The effects of turbulence on flame propagation are investigated in these limits, and identification is made of the specific parameter whose value determines which of the two limiting cases is the better approximation.

This parameter, which is essentially the ratio of the laminar flame thickness to a Taylor-like microscale for the decay of the one-point streamwise velocity-concentration correlation, differs from the simplest interpretation that can be ascribed to Damköhler's original parameter and also from Klimov's parameter γ, which in essence had earlier been introduced by Kovasznay (1956). Kovasznay interpreted γ as the ratio of the turbulent vorticity (more precisely, root-mean-square velocity fluctuation divided by Taylor microscale) to the velocity gradient in a laminar flame, a view which differs from Klimov's physical interpretation.

† There are, of course, many differences: Corrsin utilizes both Eulerian and Lagrangian viewpoints while the present study employs only the Eulerian viewpoint. Corrsin considers both early and late stages of decay of heat-transfer and r.m.s. temperature fluctuation fields subjected to a constant mean temperature gradient, while the present analysis pays little heed to the ambient stage of decay, but instead focuses attention on the dynamical growth and decay of the turbulent mass-transfer field when it is subjected to the complicated mean concentration field extant in a flame. The similarity lies in the character of the objective more than in the subject matter.

There is no logical contradition in the fact that our parameter differs from γ, since Klimov's analysis treats individual laminar flamelets within the turbulent flame while the present analysis treats the ensemble of these flamelets.

The present theory demonstrates that in the limit of large-scale turbulence, the average flame structure and the turbulent flame speed can be calculated relatively easily. Results are given for the turbulent flame speed, flame thickness and flame structure in the large-scale limit for first-order reactions without heat release. The modifications that would be involved in carrying out a corresponding calculation for finite heat release and arbitrary reaction orders are indicated. These modifications do not complicate the analysis too severely, since instead of being an eigenvalue of two simultaneous ordinary differential equations, the flame speed becomes an eigenvalue of k simultaneous ordinary differential equations, where k is a finite number whose value depends on the reaction order and on the activation energy. Thus, in some respects the approach introduced here appears promising.

2. Statement of the problem

We shall consider a particular system which has often been set up in the laboratory (see figure 1). A steady plane one-dimensional, low-speed flow of a premixed combustible fluid passes through a plane turbulence-producing grid oriented normal to the flow direction. At distances greater than roughly 10-grid mesh dimensions downstream from the grid, the mean flow is again steady and

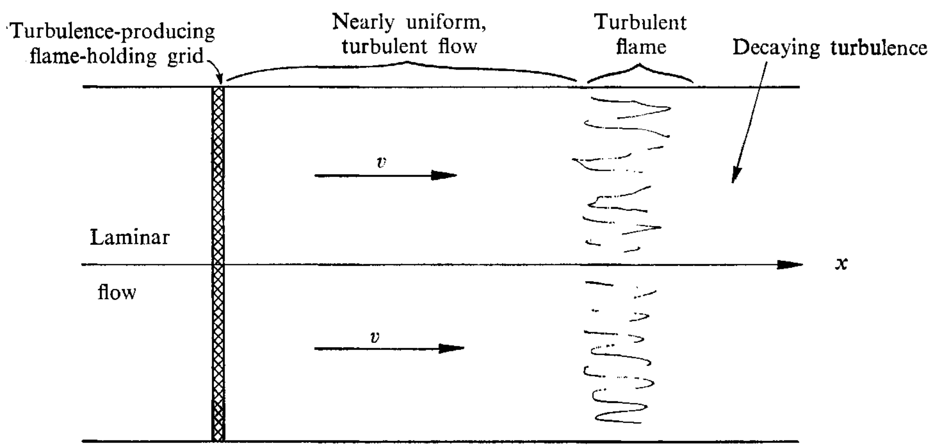

FIGURE 1. Schematic diagram of model.

one-dimensional, and if the fluid were unicomposition and non-reacting, then numerous characteristics of the fluctuating turbulent velocity field would be known from previous experimental and theoretical investigations. We shall assume that chemical reactions proceed to a negligible extent in this distance, so that except for questions concerning the production of composition and temperature fluctuations by the grid, the character of the turbulent field at about 10 or more mesh dimensions downstream may be taken as well established.

A turbulent flow, with properties that vary slowly in the flow direction through decay of turbulence, thus approaches a chemical reaction region identifiable as a steady plane one-dimensional turbulent flame.

It is well known that for laminar flames with one-step Arrhenius rate kinetics, no finite solution exists for the laminar burning velocity (the velocity at which the combustible material must flow into the flame for the position of the flame to remain stationary in space) unless the governing equations or boundary conditions are modified by introducing one of a number of artifices based upon physical understanding of the flame propagation process; e.g. see Williams (1965, chapter 5). One such artifice is the concept of an upstream flame holder that extracts heat from the fluid and serves to 'stabilize' the flame; for a reaction whose over-all activation energy is not too small, very low (but not zero) amounts of heat extraction by the flame holder lead to calculated burning velocities that are finite and are insensitive to the precise value of the rate of heat extraction. In the system under study, the turbulence-producing grid can conveniently serve a dual purpose, acting secondarily as a flame holder when the kinetic model requires an artifice to make the flame speed finite.

We wish to obtain the turbulent flame speed, which is defined as the steady-state value of the mean velocity $\bar{v}$ in the flow (x) direction that must exist in the nearly uniform turbulent flow approaching the flame for the mean position of the flame to remain stationary. In view of the fact that there exists a laminar flame speed uniquely dependent upon the thermodynamic, thermochemical, transport and chemical-kinetic properties of the system, it seems reasonable that there should exist a turbulent flame speed uniquely dependent upon these same properties plus the statistical properties of the turbulent field.

Knowledge of the turbulent flame speed is vital for predicting the performance (e.g. heat release rates, critical chamber lengths, etc.) of practical combustion equipment. However, it may be worth emphasizing that analysis of the system illustrated in figure 1 will not necessarily yield correct turbulent flame speeds for systems in which the flame does not propagate normal to the direction of the mean flow. For example, a flame spreading obliquely from a bluff body flame holder in a turbulent flow may exhibit a mean velocity component normal to the approach flow that differs from the flame speed obtained with the present model (for a flow with the same turbulent characteristics) because of a different type of flame-induced modification in the turbulent fields resulting from the non-planar geometry. The present model has been adopted primarily because of its geometrical simplicity and secondarily because it is representative of some situations encountered in practice. It seems unlikely that the understanding necessary for obtaining turbulent flame speeds in systems of more complex geometries can be developed without first developing a correspondingly deep understanding of the system illustrated in figure 1. Symmetries implied by figure 1 are discussed in appendix A.

A relevant experimental question that arises is whether the turbulent flame speed defined here exists. Turbulent flames have often been stabilized behind grids, but it is not entirely clear that these flames are not more or less attached to the grids by fingers extending upstream in the wakes. The grid may too literally act as a flame holder, in that each bar of the grid may separately hold a flame. If this occurs, then it is questionable whether a truly statistical situation can be established before the flame develops. Ensemble analyses may not be relevant for determining the flame speed. The validity of the viewpoint proposed in this paper can be tested experimentally by passing the turbulent combustible flow downstream from the grid through a slowly diverging channel, causing the mean velocity $\bar{v}$ to decrease gradually with increasing x. If a turbulent flame speed as defined herein exists, then the turbulent flame should remain stationary at the value of x where $\bar{v}$ equals the flame speed. This experiment has not been performed. Therefore, we assume that the flame speed exists experimentally and proceed to consider its mathematical existence.

3. Existence of a turbulent flame speed

The manner in which the laminar flame speed appears as an eigenvalue of the conservation equations is summarized in appendix B. Here we consider whether the turbulent flame speed can be defined in a corresponding manner, i.e. we search for equations similar to (B 1) and (B 2).

The species conservation equation for a representative reactant, whose mass fraction is Y, is

$$\frac{\partial}{\partial t}(\rho Y)+\nabla.(\rho \mathbf{v} Y)-\nabla.(\rho D\nabla Y) = w, \tag{1}$$

where ρ is density, $\mathbf{v}$ is velocity, D is a multicomponent diffusion coefficient and w is the mass rate of production of the reactant in question. The quantities ρ, $\mathbf{v}$, D and w are related to Y through other conservation equations, the state equations, expressions for transport properties, and the phenomenological chemical kinetic equations. We shall let a superior bar denote an ensemble mean and a prime identify the departure of a variable from its ensemble mean value. Taking an ensemble average of (1) yields

$$\frac{\partial}{\partial t}(\overline{\rho}\,\overline{Y})+\frac{\partial}{\partial t}(\overline{\rho' Y'})+\nabla.(\overline{\rho}\overline{\mathbf{v}}\overline{Y})+\nabla.[\overline{\rho}(\overline{\mathbf{v}' Y'})+\overline{\mathbf{v}}(\overline{\rho' Y'}) + \overline{Y}(\overline{\rho'\mathbf{v}'})]+\nabla.(\overline{\rho'\mathbf{v}' Y'})-\nabla.[(\overline{\rho D})\nabla\overline{Y}]-\nabla.[\overline{(\rho D)'\nabla Y'}] = \overline{w}. \tag{2}$$

Since means are independent of time in stationary turbulence and since one-point vector means must parallel the axes of symmetry in axi-symmetric turbulence (see appendix A), (2) reduces to

$$\frac{d}{dx}\left\{\overline{\rho}\overline{v}\overline{Y}+\overline{\rho}(\overline{v' Y'})+\overline{v}(\overline{\rho' Y'})+\overline{Y}(\overline{\rho' v'})+(\overline{\rho' v' Y'})-\left[\overline{(\rho D)'\frac{dY'}{dx}}\right]-(\overline{\rho D})\frac{d\overline{Y}}{dx}\right\} = \overline{w}, \tag{3}$$

where v' is the x component of $\mathbf{v}'$.

If the continuity equation

$$\partial\rho/\partial t+\nabla.(\rho\mathbf{v}) = 0 \tag{4}$$

is subjected to an ensemble average and use is made of the stationary and axi-symmetric implications cited above, it is found that

$$\overline{\rho}\overline{v}+(\overline{\rho' v'}) \equiv \dot{m} = \text{constant}. \tag{5}$$

Utilizing (5) in (3) yields

$$\dot{m}\frac{d\bar{Y}}{dx}-\frac{d}{dx}\left\{[(\rho D)_T+(\overline{\rho D})]\frac{d\bar{Y}}{dx}\right\} = \bar{w}, \tag{6}$$

where the turbulent mass diffusivity has been defined as

$$(\rho D)_T \equiv \left\{-\bar{\rho}(\overline{v'Y'})-\bar{v}(\overline{\rho' Y'})-(\overline{\rho' v' Y'})+\left[\overline{(\rho D)'\frac{dY'}{dx}}\right]\right\}\Big/\frac{d\bar{Y}}{dx}. \tag{7}$$

The definition of $(\rho D)_T$ appearing in (7) is rather complex and may often be useless for accurate calculations. The reason for the manipulation stems from the fact that a complete solution to the problem would yield $(\rho D)_T$ as a function of x. Hence it follows that (6) formally resembles (B 1), and that imposing the boundary conditons (B 2) on $\bar{Y}$ may therefore yield a solution for $\bar{Y}(x)$ only for a particular eigenvalue $\dot{m}$. This value would then provide a turbulent flame speed determined in a manner analogous to the way in which the laminar flame speed is determined.

The preceding development neither proves that $\dot{m}$ is an eigenvalue of (6) nor provides a method for calculating $\dot{m}$ accurately, since all of the moment problems of turbulence remain [see (7)]. However, if $(\rho D)_T$, $(\overline{\rho D})$, and $\bar{w}$ could be estimated in terms of $\bar{Y}$ and x, then the same methods that are employed in laminar flame theory could be utilized in (6) to obtain $\dot{m}$. One might venture to state that some sort of physical estimate for $(\overline{\rho D})$, $(\rho D)_T$ and $\bar{w}$ can always be made. Thus, it may not be too poor an approximation to replace $(\overline{\rho D})$ by its average laminar value, neglect $(\rho D)'$, $(\overline{\rho' v' Y'})$ and $(\overline{\rho' Y'})$, estimate $(\overline{v' Y'})$ from the relatively well-known properties of the turbulence a short distance downstream from the grid, and replace $\bar{w}$ by its laminar value $w(\bar{Y})$ plus a correction term estimated from the properties of the turbulence a short distance downstream from the grid. A very simple set of approximations of this kind is $(\overline{v' Y'}) = (\overline{v'^2})^{\frac{1}{2}}$, $(\rho D)' = 0$, $(\overline{\rho' v' Y'}) = 0$, $(\overline{\rho' Y'}) = 0$, $\bar{w} = w(\bar{Y})$. The proper question therefore appears not to be whether the turbulent flame speed problem can be cast as an eigenvalue problem but rather how accurately the turbulent flame speed can be calculated when the problem is so cast. In the following sections we investigate this question for a system in which most of the complexities are eliminated.

4. A turbulent flame with vanishing heat release and with a first-order reaction

To begin a more thorough investigation of the properties of (6), suppose there is only one reactant, the reaction is of first order with respect to this reactant, and the heat release is small compared with the initial thermal enthalpy of the system, so that the density may be assumed to be constant. Although there exist chemical reactions for which these assumptions are approximately valid, these reactions are of little or no interest in combustion, and therefore the present treatment is best considered to represent an attempt to uncover a few of the elements that must appear in analyses of realistic flames. The analysis is not trivial even under these highly restrictive assumptions.

The assumptions concerning the reaction rate reduce (1) to a linear equation in Y and enable us to write w in the form

$$w = -w_0 Y, \tag{8}$$

where w_0 is a positive constant. For this reaction rate function a pseudostationary value of the laminar flame speed does not exist, and the calculated flame speed will depend upon the value assigned to Y_0' in (B 2). However, this undesirable aspect of the hypotheses does not necessarily affect the relationship between the laminar and turbulent flame speeds. A useful simplification in the turbulent flame analysis produced by (8) is that $\overline{w(Y)} = w(\overline{Y})$.

In addition to eliminating three terms in (7), the assumption concerning the heat release (ρ = constant) has the extremely useful effect of decoupling the equation for the velocity field from the equation for the concentration field. The statistics of the velocity field may therefore be assumed known in studying the turbulent flame speed problem, and we shall investigate the properties of a concentration field imposed on a known velocity field, in an attempt to ascertain what the mean flow velocity must be for the concentration field to possess a solution satisfying turbulent flame boundary conditions.

Finally, we shall also assume that the diffusion coefficient D is constant, and we shall equate D to the kinematic viscosity (i.e. the Schmidt number will be taken to be unity). The latter approximation is very good for gases, and the former is consistent with the assumption of small heat release.

Under the stated assumptions, (1) becomes

$$\frac{\partial Y}{\partial t} + \bar{v}\frac{\partial Y}{\partial x} + \nabla . (\mathbf{v}' Y) - D\nabla^2 Y = -(w_0/\rho)\, Y, \tag{9}$$

the continuity equation is
$$\nabla . \mathbf{v}' = 0, \tag{10}$$

and the momentum conservation equation becomes

$$\frac{\partial \mathbf{v}'}{\partial t} + \bar{v}\frac{\partial \mathbf{v}'}{\partial x} + (\mathbf{v}' . \nabla)\,\mathbf{v}' = -\nabla(p/\rho) + D\nabla^2 \mathbf{v}', \tag{11}$$

where $\bar{v}$ is the constant mean velocity in the x direction, $\mathbf{v}'$ is the difference between the vector velocity and its mean value, and p is pressure. As may easily be inferred from (5), (6) and (7) under the current set of restrictions, an ensemble average of (9) yields

$$\bar{v}\frac{d\overline{Y}}{dx} - D\frac{d^2\overline{Y}}{dx^2} + \left(\frac{w_0}{\rho}\right)\overline{Y} = -\frac{d}{dx}(\overline{v' Y'}), \tag{12}$$

where v' is the x component of $\mathbf{v}'$ and $Y' \equiv Y - \overline{Y}$. The only term in (12) that is not present for laminar flow is the one involving $\overline{v' Y'}$, which is the diffusional analogue of a Reynolds stress since $\rho\overline{v' Y'}$ represents the mean value of the mass of reactant per unit area per second transported by the turbulent fluctuations across a plane normal to the x axis and moving with velocity $\bar{v}$ in the x direction. Equation (12) cannot be solved for the turbulent burning velocity unless $(\overline{v' Y'})$ can be determined in some manner.

An equation for $(\overline{v'Y'})$ can be obtained by the classical approach of multiplying the equation for Y' by v', multiplying the equation for v' by Y', adding the results together and averaging. The equation for Y' is

$$\frac{\partial Y'}{\partial t}+\bar{v}\frac{\partial Y'}{\partial x}+v'\frac{d\bar{Y}}{dx}-\frac{d}{dx}(\overline{v'Y'})+\nabla.(\mathrm{v}'Y')-D\nabla^2Y' = -\left(\frac{w_0}{\rho}\right)Y', \tag{13}$$

which may be obtained by subtracting (12) from (9) and employing (10). The equation for v' is the x component of (11). The operations stated above yield

$$\bar{v}\frac{d}{dx}(\overline{v'Y'})-D\frac{d^2}{dx^2}(\overline{v'Y'})+\left(\frac{w_0}{\rho}\right)(\overline{v'Y'}) = -\overline{v'^2}\frac{d\bar{Y}}{dx}$$

$$-2D\overline{[(\nabla v').(\nabla Y')]}-\frac{d}{dx}(\overline{v'^2Y'})-\left[\overline{Y'\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)}\right], \tag{14}$$

where symmetry conditions have been employed. As expected, (14) for $(\overline{v'Y'})$ contains (on the right-hand side) terms involving means other than $(\overline{v'Y'})$ and $\bar{Y}$. However, certain observations can be made concerning these terms.

5. Properties of mass-flux equation

The first term on the right-hand side of (14) causes no analytical difficulty under our present assumptions because $\overline{v'^2}$ is a known function of x when the velocity field is known. This term accounts for production of $(\overline{v'Y'})$ by the mean concentration gradient.

Terms of the form of the second term on the right-hand side of (14) appear in analyses of homogeneous, isotropic turbulent fields; they account for viscous dissipation [dissipation of $(\overline{v'Y'})$ in the present case] and are often written as the ratio of the quantity being dissipated to the square of a characteristic length. We shall therefore introduce as a definition

$$\overline{[(\nabla v').(\nabla Y')]} \equiv (\overline{v'Y'})\,(l_x^{-2}+2l_\perp^{-2}), \tag{15}$$

in which l_x and $l_\perp$ are characteristic lengths in the x direction and in a direction normal to x, respectively (explicitly,

$$l_x \equiv \left[(\overline{v'Y'})\Big/\left(\overline{\frac{\partial v'}{\partial x}\frac{\partial Y'}{\partial x}}\right)\right]^{\frac{1}{2}}$$

and similarly for $l_\perp$). Since the x direction is special, it is necessary to allow for the possibility of two different lengths in the present problem. Since l_x and $l_\perp$ are permitted to be functions of x, (15) is exact by definition; we have merely transferred the problem of computing the left-hand side of (15) to the problems of computing l_x and $l_\perp$. We note, however, that unlike corresponding lengths for $(\overline{v'^2})$ and $(\overline{Y'^2})$, there is no guarantee that l_x^2 and $l_\perp^2$ will be positive. It is possible for conditions to occur under which viscous-diffusive effects tend to increase the magnitude of $(\overline{v'Y'})$, causing l_x and/or $l_\perp$ to be imaginary. Nevertheless, (15) may give an indication of the order of magnitude of the dissipation term, because

l_x and $l_\perp$ might be expected to be roughly of the order of the Taylor microscale of the velocity field.

The last two terms in (14) both involve means of products of three fluctuating quantities, each of which has a zero mean value, the first term manifestly so and the second term through (10) and (11).† One might note that in the present problem, p/ρ is unaffected by the flame; it is a property of the velocity field which is presumed known and which (being grid-produced turbulence) empirically is inferred to exhibit a pressure–velocity relationship that does not differ greatly from that of homogeneous turbulence. Although the last two terms in (14) exert a profound influence on the dynamics of the decay of turbulent eddies, by comparing the next-to-last term with the first term on the left-hand side it is readily inferred that the last two terms are small in this particular equation provided that $v' \ll \bar{v}$. The flame-produced changes in $(\overline{v'Y'})$, introduced through $d\overline{Y}/dx$, outweigh the turbulent spectral transport terms. It therefore does not appear to be grossly unreasonable to neglect the last two terms in (14) provided that a representative fluctuation velocity, e.g. the turbulence intensity $(\overline{v'^2})^{\frac{1}{2}}$, is small compared with the mean velocity $\bar{v}$, as is generally true for grid turbulence.

If we assume that the fluctuation velocity is small compared with the mean velocity, then (14) can be written as

$$\bar{v}\frac{d}{dx}(\overline{v'Y'}) - D\frac{d^2}{dx^2}(\overline{v'Y'}) + \left(\frac{w_0}{\rho}\right)(\overline{v'Y'}) + \frac{6D}{\bar{l}^2}(\overline{v'Y'}) = -\overline{v'^2}\frac{d\overline{Y}}{dx}, \tag{16}$$

where

$$\bar{l} \equiv [3(l_x^{-2} + 2l_\perp^{-2})^{-1}]^{\frac{1}{2}}. \tag{17}$$

Since $\overline{v'^2}$ is known, (16) becomes a linear equation involving only $(\overline{v'Y'})$ and $\overline{Y}$ as unknown functions, provided that $\bar{l}$ is assumed known. Equations (12) and (16) then constitute two coupled linear ordinary differential equations which, under the present set of assumptions, should serve to determine the value of the turbulent flame speed $\bar{v}$ when appropriate boundary conditions are imposed. The turbulent flame speed thus appears as an eigenvalue of two coupled equations instead of the single equation obtained in the laminar case. It will be noted, however, that many restrictions were needed and much of the dynamics of the turbulence had to be suppressed in order to arrive at a prescription for computing the turbulent flame speed at the present low level of complexity.

6. Parameter defining limits of turbulent scale

Since triple correlations are negligible only when the turbulence intensity is sufficiently low for the effect of the turbulence on the flame to be small, in (16) $\bar{v}$ will be of the same order of magnitude as v_L and the characteristic distance over which quantities involving Y change appreciably will be of the order of δ_L, where v_L and δ_L are defined in appendix B. Hence the first three terms in (16)

† The divergence of (11) yields $\nabla^2(p/\rho) = -\nabla . [(\mathbf{v}' . \nabla)\, \mathbf{v}']$ when use is made of (10). This implies that p/ρ can be related to the space integral of a function involving $\mathbf{v}'$ quadratically.

are all of the order of $(D/\delta_L^2)\,(\overline{v'Y'})$. Two limiting cases can then be distinguished in (16): if $\delta_L \gg |\bar{l}/\sqrt{6}|$, then (16) reduces approximately to

$$(\overline{v'Y'}) = -\frac{\overline{l^2 v'^2}}{6D}\frac{d\overline{Y}}{dx}, \tag{18}$$

while if $\delta_L \ll |\bar{l}/\sqrt{6}|$, then the term involving $\bar{l}$ is negligible and (16) becomes

$$\bar{v}\frac{d}{dx}(\overline{v'Y'}) - D\frac{d^2}{dx^2}(\overline{v'Y'}) + \left(\frac{w_0}{\rho}\right)(\overline{v'Y'}) = -\overline{v'^2}\frac{d\overline{Y}}{dx}. \tag{19}$$

The transition régime, in which (16) must be used instead of either (18) or (19), occurs in the neighbourhood of $\kappa = 1$, where

$$\kappa \equiv \sqrt{6}\,\delta_L/|\bar{l}|. \tag{20}$$

It can be seen from (15) and (17) that the condition $\kappa = 1$, separating the régimes of large-scale and small-scale turbulence, can be expressed in the form

$$2\delta_L^2|\langle[\overline{(\nabla v').(\nabla Y')}]/(\overline{v'Y'})\rangle| = 1,$$

where the angular brackets imply a mean value in the flame zone. This indicates that the length $|\bar{l}/\sqrt{6}|$ to which δ_L is equal in the transition region is roughly interpretable as the square root of the product of a length characteristic of the gradient of the velocity fluctuation field and a length characteristic of the gradient of the reactant concentration fluctuation field. The condition $\kappa = 1$ therefore differs from that of Damköhler viz. $\delta_L = d$, where d is a characteristic eddy dimension, and also from that of Kovasznay and Klimov, viz. $\gamma = 1$, which according to Kovasznay's interpretation can be expressed approximately as

$$(\delta_L^2/v_L^2)\langle[\overline{(\nabla v').(\nabla v')}]\rangle/5 = 1$$

in the present notation. The present criterion differs conceptually from both of the earlier ones in that the concentration field is elevated to a status equal to that of the velocity field. It differs further from that of Kovasznay and Klimov in that v_L does not appear.

A numerical comparison of the three turbulent scale criteria cannot be given properly because of the imprecision of Damköhler's definition of eddy dimension and because of absence of information concerning the dynamics of the turbulent concentration field which appears in the present criterion. Merely to obtain some sort of comparison, let us assume that Y' can be replaced by v' in our criterion. Let us also define Damköhler's eddy dimension d by the equation for the rate of decay of kinetic energy,

$$(\overline{v'^2})^{\frac{3}{2}}/d = D[\overline{(\nabla v').(\nabla v')}],$$

so that d approximately equals the reciprocal of a wave-number characteristic of the energy-containing part of the spectrum. When the relationship $v_L = D/\delta_L$, which follows from (B 3) and (B 4), is used to eliminate v_L from the criterion of

Kovasznay and Klimov, then simple algebra produces the following comparison of the three criteria:

$$\left.\begin{aligned} &\text{Damköhler:} && \delta_L = d, \\ &\text{Kovasznay–Klimov:} && \delta_L = 5^{\frac{1}{4}} d/R^{\frac{3}{4}}, \\ &\text{present study:} && \delta_L = d/(2R)^{\frac{1}{2}}. \end{aligned}\right\} \tag{21}$$

Here the Reynolds number of the turbulence is defined as $R \equiv (\overline{v'^2})^{\frac{1}{2}} d/D$. At $R = 100$, (21) shows that the critical value of d/δ_L is 1 for Damköhler's criterion, 21 for the Kovasnay–Klimov criterion, and 14 for the new criterion. Thus, the present criterion would appear to lie between the two earlier ones, numerically and also in respect to the functional dependence on R. For representative values of R, the present criterion appears to lie closer to that of Kovasznay and Klimov. On the other hand, if Damköhler's eddy dimension were taken to be $|\bar{l}/\surd 6|$ then his criterion would agree with the present one. The comparisons are not entirely clear-cut. Ignorance of the dynamics of the composition fluctuation field prevents accurate comparisons from being made for the present criterion.

7. Limit of small-scale turbulence

Although precise criteria are not available for defining the range of validity of (18) or (19), the limits nevertheless exist. It is therefore of interest to examine the nature of the solutions in these limits more thoroughly.

In the limit of small-scale turbulence, substitution of (18) into (12) yields

$$\bar{v}\frac{d\bar{Y}}{dx} - \frac{d}{dx}\left[\left(D + \frac{\bar{l}^2\overline{v'^2}}{6D}\right)\left(\frac{d\bar{Y}}{dx}\right)\right] + \left(\frac{w_0}{\rho}\right)\bar{Y} = 0. \tag{22}$$

The pair of coupled differential equations governing the turbulent flame speed therefore reduces to a single differential equation which in fact is of exactly the same form as the differential equation governing the laminar flame speed. The only difference between (22) and the corresponding equation for laminar flow is that in place of the laminar diffusion coefficient D, the effective diffusion coefficient $D + \bar{l}^2\overline{v'^2}/6D$ appears. The second term in this expression represents the increase in the effective diffusion coefficient caused by turbulence; the value of $\overline{v'^2}$ appearing there is a known function of x under the present assumptions, but the value of $\bar{l}$ will be known only near $x = 0$ since methods for obtaining the x dependence of $\bar{l}$ have not been considered. Nevertheless, the effect of turbulence on the flame thickness and the flame speed may be estimated by replacing the product $\bar{l}^2\overline{v'^2}$ by a constant average value in the flame $\langle\bar{l}^2\overline{v'^2}\rangle$ which may, for example, be roughly approximated as the value near $x = 0$. The same reasoning that leads to the laminar expressions for δ_L and v_L then shows us that

$$\delta_T/\delta_L = \bar{v}_T/v_L = \surd(1 + \langle\bar{l}^2\overline{v'^2}\rangle/6D^2), \tag{23}$$

where δ_T is the thickness of the turbulent flame and $\bar{v}_T$ is the turbulent burning velocity.

Aside from the fact that $\bar{l}$ is not really known, (23) represents an explicit expression for the change in the flame thickness and the change in the flame speed caused by small-scale turbulence. We expect physically that the average value of $\bar{l}^2$ in the flame will be positive and therefore that small-scale turbulence increases both the flame thickness and the flame speed. Damköhler's deduction that turbulence principally modifies the effective diffusivity in this limiting case is placed on a firmer basis for the special system under investigation by the present result, which also provides a formula (viz. $\rho\bar{l}^2\overline{v'^2}/6D$) for the appropriate turbulent diffusivity for this system. However, certain reservations should be raised about the result. The approximation to (14), expressed by (18), constitutes a balance between production of $(\overline{v'Y'})$ (through gradients in the mean concentration field) and viscous-diffusive decay of $(\overline{v'Y'})$ (for the usual case in which $(\overline{v'Y'})$ is positive). Although the triple correlation terms in (14) superficially appear to be of higher order, to have the direct interchange between production and dissipation without intervening spectral transfer, as implied by (18), seems somewhat questionable on physical grounds unless the value of the Reynolds number is too small to be very interesting. This troublesome aspect, which can be resolved only through a more thorough investigation of $\bar{l}$, does not arise in the limit of large-scale turbulence.

8. Character of the limit of large-scale turbulence

For the limiting case in which (19) is valid, a number of different turbulent burning velocity formulas have been obtained from wrinkled flame theories, based on different assumptions concerning the geometry of wrinkling and its effects. Equations (12) and (19) can help in evaluating the bases of these analyses. Results obtained from these two linear equations will be relatively free from objection since $\bar{l}$ does not appear. According to (19), the production of $(\overline{v'Y'})$ in the flame is manifest directly in the streamwise change of this quantity in the flame environment. Residence times in the thin flame zone are too short for dissipation or spectral transfer to be of significance. The situation bears some resemblance to Batchelor's (1953) problem of a turbulent field passing through a sudden contraction. Since $\overline{v'^2}$ is approximately constant across the flame under the present conditions, (12) and (19) can be solved relatively easily for the turbulent flame speed and for the average flame structure. This is fortunate because the estimates discussed in § 6 suggest that (12) and (19) possess an appreciably large range of validity ($\kappa \ll 1$), somewhat larger than the range of the Kovansznay–Klimov condition $\gamma \ll 1$ for reasonable values of the Reynolds number. It is of interest to discuss some qualitative aspects of the solution before carrying out the calculation.

To guess the relationship between the laminar and turbulent flame speeds predicted by (12) and (19), one might try neglecting the second and third terms in (19) compared with the first. One then obtains the single equation

$$\left(\bar{v}-\frac{\overline{v'^2}}{\bar{v}}\right)\frac{d\bar{Y}}{dx}-D\frac{d^2\bar{Y}}{dx^2}+\left(\frac{w_0}{\rho}\right)\bar{Y}=0,$$

which implies that $[\bar{v}-(\overline{v'^2}/\bar{v})]$ for turbulent flow has the same value as v for laminar flow; i.e.

$$\bar{v}_T \approx v_L+\overline{v'^2}/v_L. \tag{24}$$

This result bears some resemblance to formulas obtained from wrinkled laminar flame theories; it looks most like a formula obtained by Tucker (1956).

However, there is no reason to assume that the first term in (19) is dominant. If one assumes that the first two terms in (19) can be neglected in comparison with the third, then one obtains from (12) an equation of the form of (22) with the turbulent diffusion coefficient given by $\overline{v'^2}/(w_0/\rho) = \delta_L^2\overline{v'^2}/D$; the physical interpretation is then the same as for small-scale turbulence (except for the modified formula for the turbulent diffusivity), the only effect of turbulence being to increase the effective diffusion coefficient. One obtains

$$\bar{v}_T \approx v_L\sqrt{(1+\delta_L^2\overline{v'^2}/D^2)} \tag{25}$$

and a similar expression for the increase in the flame thickness.

On the other hand, one might assume that the second term in (19) dominates the left-hand side. The equation derived from (12) and (19) is then somewhat more complicated, but if one assumes that the change in $\overline{v'^2}$ is negligible across the turbulent flame, then one obtains an equation that is exactly the same as the equation governing the laminar flame speed except that the reaction rate term $(w_0/\rho)\bar{Y}$ is replaced by the term $[(w_0/\rho)+(\overline{v'^2}/D)]\bar{Y}$.† The turbulence would do nothing but increase the effective reaction rate, thereby yielding (25) for the increase in the turbulent flame speed, but actually exhibiting a decrease in the flame thickness due to turbulence.

Clearly, none of these simplifications are correct. All terms in (19) must be retained, and two coupled equations must be solved to obtain the flame speed, as is done in appendix C, and discussed in the following section.

9. Flame speed in the limit of large-scale turbulence

Equation (C 12) shows that the fractional change in flame speed is proportional to the turbulence intensity $\epsilon = \overline{v'^2}/\bar{v}^2$, with a constant of proportionality which depends on the laminar flame speed (through β_L), on the initial turbulent correlation between velocity and concentration fields (through g_0), and on any change in the concentration-gradient parameter β produced by turbulence (through β_1).

Since β is a rather non-physical parameter, the implications of the last term in (C 12) are difficult to interpret. There is no good way to determine either the sign or the magnitude of β_1. This result constitutes a drawback of the physical model adopted herein. We find the drawback to be especially appalling when we note that if β_1 is large enough then the sign of $(\bar{v}_T - v_L)/v_L$ depends on the sign of β_1; turbulence is found to *decrease* the flame speed if β_1 is both positive and sufficiently large. It is necessary to introduce more physics in order to make statements about β_1. In the corresponding laminar models, it is well known that β can be interpreted as a heat loss (or a loss of a chemical species by diffusion) to a flame holder. If the loss occurs primarily in a convective process for which the

† An additive constant is set equal to zero because $Y \to 0$ [i.e. $\bar{Y} \to 0$ and $Y' \to 0$, implying $(\overline{v'Y'}) \to 0$] as the reaction goes to completion at the downstream boundary of the flame where the velocity field and $\overline{v'^2}$ have changed negligibly.

characteristic velocity is $\bar{v}$, then one might expect the loss, in physical coordinates, represented by the physical gradient Y_0', to be proportional to $\bar{v}$. If this assumption is correct, then β is the same for the laminar and turbulent problems, and consequently $\beta_1 = 0$. In the following discussion the value of β_1 will be taken to be zero.

The value of the velocity–concentration correlation term g_0 at the grid or flame holder can also affect the sign of $(\bar{v}_T - v_L)/v_L$. If g_0 is sufficiently large and positive, then $(\bar{v}_T - v_L)/v_L$ can become negative. If $\beta_1 = 0$, the condition for $(\bar{v}_T - v_L)/v_L$ to be positive can be expressed as $g_0 < (1+\beta_L)/(1+2\beta_L)^2$, the left-hand side of which is a monotonically decreasing function of β_L which goes from 1 at $\beta_L = 0$ to 0 at $\beta_L = \infty$. Positive values of g_0 correspond to excess turbulent velocities in the downstream direction being accompanied by increases in the reactant concentration. The magnitude and sign of g_0 depend on characteristics of the turbulence-producing grid. If the grid produces isotropic turbulence, then since $(\overline{v'Y'})$ is a component of a vector quantity, it follows that $(\overline{v'Y'})_0 \equiv 0$, and consequently $g_0 = 0$. Physically, it would seem that a non-catalytic grid would be likely to produce (non-isotropic) velocity fluctuations without producing any concentration fluctuations at all; in this case the value of the quantity g_0 would again be zero. Thus, the choice $g_0 = 0$ appears to be reasonable. We might emphasize, however, that even if $g_0 = 0$, the solution for $g(\eta)$ given in (C 9) shows that g becomes positive in the flame zone; thus, the flame may be said to generate a positive correlation between the streamwise velocity field and the reactant concentration field. The flame itself enhances turbulent mass transport.

When $\beta_1 = 0$ and $g_0 = 0$, (C 12) assumes a very simple form, which can be written in terms of physical quantities as

$$(\bar{v}_T - v_L)/v_L = \overline{v'^2}/(2v_L^2 + 8Dw_0/\rho). \tag{26}$$

This formula is not exactly the same as any of the three simplified results discussed in the preceding section; it appears to predict an increase in burning velocity which is of about the same order of magnitude as, but somewhat smaller than, the increase predicted by any of the three preceding simplified results. The predicted increase also appears to be smaller than predictions of earlier investigators; its functional form resembles that of Shelkin (1943) or that of Tucker (1956) somewhat more closely than it resembles any other previous results.

10. Average flame structure in the limit of large-scale turbulence

A turbulence-produced modification in the structure of the average reactant concentration field is evident in (C 4) and (C 10). We may first note from (C 4) that if f_1 were equal to zero, then the definition of η would cause the physical distance over which $\bar{Y}$ experiences any specified change to be less for turbulent flow than for laminar flow, provided that $\bar{v}_T > v_L$. Thus, this scale effect tends to cause a turbulent flame to be thinner than the corresponding laminar flame. However, at least when $\beta_1 \leqslant 0$, the contribution of f_1 tends to counteract the scale effect and to make the turbulent flame thicker. Further investigation is needed to ascertain which of the two opposing effects is greater.

The comparison between laminar and turbulent flame structure can be facilitated by employing the laminar non-dimensional co-ordinate $\eta_L \equiv xv_L/D$ in the turbulent-flame formula for the x-distribution of $\bar{Y}$. The result (to first order in $\epsilon_L \equiv \overline{v'^2}/v_L^2$) for $f = \bar{Y}/Y_0$ in the turbulent flame is readily found to be

$$f = \left[1 + \frac{\epsilon_L \beta_L \eta_L(\beta_L \eta_L - 1)}{2(1+2\beta_L)^2}\right] e^{-\beta_L \eta_L}, \tag{27}$$

when $\beta_1 = 0$. This result is to be compared with the laminar reactant distribution given in (C 4). It is seen that at small values of $\beta_L \eta_L$ (viz. for $\eta_L < 1/\beta_L$) the turbulent f is less than f_L, while at large values of $\beta_L \eta_L$ (viz. for $\eta_L > 1/\beta_L$) the turbulent f is greater. At $\eta_L = 1/\beta_L$, the laminar and turbulent values of f are always identical. The result is illustrated in figure 2, where f is plotted as a function of $\beta_L \eta_L$ for various values of $\epsilon_L/2(1+2\beta_L)^2$ with $\beta_1 = 0$. Turbulence enhances reactant consumption near the grid of flame holder and lessens reactant consumption downstream.

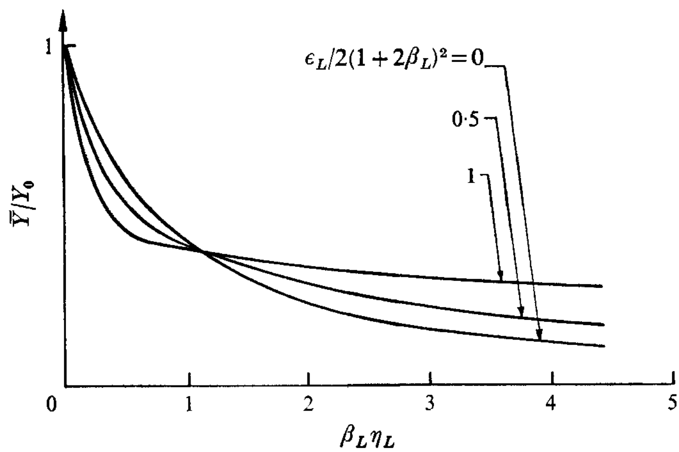

FIGURE 2. Reactant concentration profiles for various values of turbulence intensity.

It is clear from these results that if the flame thickness is defined as the value of x at which $\bar{Y}$ has decreased to e^{-n} times its initial value, then one will always find that the turbulent flame is thicker than the laminar flame provided that one selects a value of n greater than unity. Letting δ_L and δ_T denote the laminar and turbulent flame thicknesses, respectively, we easily find that

$$(\delta_T - \delta_L)/\delta_L = (n-1)\,\epsilon_L/2(1+2\beta_L)^2. \tag{28}$$

Since it appears to be most reasonable to select a value of $n \gtrsim 2$ or 3 in defining flame thicknesses, we may conclude that turbulence thickens the flame, according to the most realistic notions of flame thickness. In fact, by comparing (26) and (28), one finds that

$$(\delta_T - \delta_L)/\delta_L = (n-1)\,(\bar{v}_T - v_L)/v_L \approx (\bar{v}_T - v_L)/v_L; \tag{29}$$

the flame is thickened in proportion to the increase in flame speed.

The turbulent reactant flux distribution is given in (C 9). It is not difficult to calculate x distributions of other ensemble averages in the present limiting case. For example, a linear equation for $\overline{Y'^2}$ can easily be derived and solved. We shall not pursue these calculations.

11. Extensions to more realistic models

One objection to the analysis for large-scale turbulence is the absence of a pseudostationary value for the burning-rate eigenvalue. This result stems from the choice of the reaction-rate function. It is possible to investigate the effects of adopting a more realistic reaction-rate function by considering

$$w = w_0 Y^m (1-Y)^n.$$

Following averaging procedures analogous to those given earlier, one can conclude that in this case at least three differential equations must be retained for calculating the burning rate; the new equation is for $\overline{Y'^2}$. The equations are coupled non-linearly through terms that do not involve x derivatives. Unless m and n are both small ($\lesssim 1$), one must retain more than three equations. Equations for $\overline{Y'^3}$, $\overline{Y'^4}, \ldots, \overline{Y'^{m+n}}$ will enter, and those through $\overline{Y'^k}$ (for some definable $k < m+n$) will be important. Large values of n, 10 to 15, are of interest because the $(1-Y)^n$ factor approximates the effect of an activation energy. Thus, the complexity of this type of turbulent flame-speed calculation increases as the model is made more realistic. However, there seems to be no fundamental barrier in pursuing this line of extension. The system of coupled ordinary differential equations can be solved by available finite-difference routines on digital computers. Turbulence problems such as those associated with the determination of $\bar{l}$ do not arise.

Although we have not discussed the energy conservation equation, extensions of the type just indicated should provide qualitative information about its solution. It would also be of interest to write the proper statistical energy conservation equation for systems with non-negligible heat release and to investigate the modifications in the type of approach developed herein that would be needed to analyze it. Studies analogous to those indicated in the preceding paragraph should be able to elucidate the effect of the reaction-rate term in the energy conservation equation. The effect of the heat release on the velocity field, which will also come into the correct energy equation, is expected to be somewhat more difficult to handle.

It will probably be considerably more difficult to analyze systems with large turbulent velocity fluctuations ($\epsilon \gtrsim 1$).

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Appendix A. Symmetry

In the subject of homogeneous isotropic turbulence, geometrical symmetries afford a considerable simplification in analyses of turbulent dynamics. It is clearly of interest to inquire whether the symmetries of the present problem can be comparably helpful. From the difficulties encountered in attempting rigorously to relate incompressible, non-reacting grid-produced turbulence to incompressible, homogeneous, isotropic turbulence, it should be apparent that few useful results will be obtainable from symmetry considerations alone and that a much smaller number of symmetry relationships will hold for the present system than for a homogeneous, isotropic system. We merely point out a few consequences of the symmetry of the present problem.

The turbulence in the problem under study is *stationary* and *axisymmetric.* The first of these adjectives means that the statistical properties of the fields are independent of time. As used here, the second means only that, at any given axial position x, the statistical properties of the fields are independent of the other two Cartesian co-ordinates y and z; it does not imply invariance upon reflexion in planes normal to the x axis as is sometimes included in the definition of the term 'axisymmetric' (Chandrasekhar 1950). A consequence of these observations is that in order to initiate a theoretical study of the present problem analogous to the existing theory of homogeneous turbulence it is advisable to introduce multipoint ensemble mean values in which the y, z and t values of the points may differ but the x values of the points are all the same. Fourier transforms of the various mean quantities with respect to y, z and t can be introduced, but a transform with respect to x, which is analogous to the t co-ordinate of homogeneous turbulence, is not acceptable. It would be acceptable if we could invoke a mean-speed convection hypothesis of the type that Taylor and others used for grid turbulence, but we must retain the possibility of rapid x-wise changes in the statistical properties of the turbulent field which may be induced by a thin flame. The transform method is not introduced in this paper because it does not appear to be either necessary or helpful for our purposes.

The symmetry of the problem implies translation and reflexion invariances in y, z and t and rotation invariance in the y, z plane. These invariances impose certain restrictions on multipoint ensemble means. For example, a two-point $(x, y_1, z_1, t_1; x, y_2, z_2, t_2)$ scalar mean is a function of only the three independent variables x, $[(y_2-y_1)^2+(z_2-z_1)^2]^{\frac{1}{2}}$ and $|t_2-t_1|$, as is the x component of a two-point vector mean and the xx component of a two-point second-rank tensor mean. The y component of a two-point vector mean is the product of (y_2-y_1) with a function of the same three independent variables, and the z component is the product of (z_2-z_1) with the *same* function of the same three independent variables. This remark also applies to the xy and xz components of a two-point second-rank tensor mean and (separately) to the yx and zx components of a two-point second-rank tensor mean. The yz and zy components of a two-point second-rank tensor mean must be equal and are given by $(y_2-y_1)(z_2-z_1)$ times a function of the three independent variables indicated earlier, while the yy and zz components of a two-point second-rank tensor mean are given, respectively, by $A+(y_2-y_1)^2 B$ and $A+(z_2-z_1)^2 B$, where A and B are functions of the three independent variables identified previously. Only certain of the simplest consequences of these observations are required in the development; for example, we employ the rather obvious fact that the y and z components of a one-point vector ensemble mean must vanish.

Appendix B. The laminar flame speed

It may be appropriate to state explicitly a few basic assumptions that pervade our entire development and help to define the system under investigation. We consider only one-step reactions in ideal gas mixtures. Body forces, radiative heat transfer, energy and species losses from the system anywhere except at the grid, the Soret and Dufour effects and diffusion caused by pressure gradients are all neglected. The ensemble-mean ordered kinetic energy is neglected in comparison with the ensemble mean thermal enthalpy here and in § 3.

Under the preceding assumptions, it is known that the laminar flame speed is governed in essence by an equation resembling the non-homogeneous diffusion equation. Whether the dependent variable in this equation is temperature, thermal enthalpy, concentration of a reactant, concentration of a product, etc., depends largely on the taste of the investigator; simple relationships exist among these variables under assumptions that are not unrealistic, and even if the necessary assumptions are not imposed, so that the relationships among the variables involve differential equations, freedom in the choice of the variable employed for obtaining the flame speed still exists for realistic ideal gas mixtures. We choose the mass fraction Y of a representative reactant as the dependent variable in the equation determining the flame speed.

For steady laminar flow in the x direction, (1) reduces to

$$\dot{m}\frac{dY}{dx}-\frac{d}{dx}\left(\rho D\frac{dY}{dx}\right)=w, \tag{B 1}$$

where $\dot{m}\equiv\rho v$ (v = component of $\mathbf{v}$ in the x direction) is the (constant) mass flow rate per unit area. For a laminar flame, appropriate boundary conditions for (B 1) are

$$\left.\begin{array}{l} Y=Y_0, \quad dY/dx=Y_0' \quad \text{at} \quad x=0, \\ Y=Y_\infty \quad \text{at} \quad x=\infty, \end{array}\right\} \tag{B 2}$$

where Y_0, Y_0' and Y_∞ are specified constants.† Although it is not obvious from the equation and the boundary conditions, it follows from laminar flame theory, after ρD and w are related to Y by solving the appropriate equations, that there exists a solution to (B 1) subject to the specified boundary conditions only for a unique value of $\dot{m}$, see Williams (1965, chapter 5). This eigenvalue $\dot{m}$ is the laminar burning velocity (multiplied by the upstream density). The solution for $\dot{m}$ demonstrates that the laminar burning velocity is of the order of

$$v_L\equiv[D(w_0/\rho)]^{\frac{1}{2}}, \tag{B 3}$$

† The choice of $x=0$ as the 'cold boundary' and the specification of both Y_0 and Y_0' remove the 'cold boundary difficulty'.

and that the thickness of the laminar flame is of the order of

$$\delta_L \equiv [D/(w_0/\rho)]^{\frac{1}{2}}, \tag{B 4}$$

where ρ and D are evaluated in the unburnt mixture and where w_0 is the maximum value of w, divided by Y_0.

Appendix C. Analysis in the limit of large-scale turbulence

In terms of the dimensionless independent variable $\eta \equiv x\bar{v}/D$, and the dependent variables $f \equiv \overline{Y}/Y_0$ and $g \equiv (\overline{v'Y'})\,\bar{v}/Y_0\overline{v'^2}$, (12) and (19) become

$$\frac{df}{d\eta}-\frac{d^2f}{d\eta^2}+\Gamma f = -\epsilon\frac{dg}{d\eta}, \tag{C 1}$$

and

$$\frac{dg}{d\eta}-\frac{d^2g}{d\eta^2}+\Gamma g = -\frac{df}{d\eta}, \tag{C 2}$$

where $\Gamma \equiv Dw_0/\bar{v}^2\rho$ and $\epsilon \equiv \overline{v'^2}/\bar{v}^2$. The quantity Γ is the burning-rate eigenvalue, while ϵ measures the (known) turbulence intensity. The boundary conditions for (C 1) and (C 2) are

$$\left.\begin{array}{lll} f=1, \quad df/d\eta = -\beta \quad \text{and} \quad g = g_0 & \text{at} & \eta = 0, \\ f = g = 0 & \text{at} & \eta = \infty, \end{array}\right\} \tag{C 3}$$

where $\beta \equiv -DY_0'/Y_0\bar{v}$, and $g_0 \equiv (\overline{v'Y'})_0\bar{v}/Y_0\overline{v'^2}$ are presumed to be specified constants. The mathematical problem is to solve (C 1) and (C 2) subject to the boundary conditions given in (C 3), and to obtain the eigenvalue Γ.

To interpret the solution to this turbulent problem properly, it is necessary to have the solution to the corresponding laminar problem. The laminar problem is easily defined by setting $\epsilon = 0$ and by ignoring the equation for g. It is easy to show that the laminar solution is

$$f_L = e^{\alpha_L\eta}, \tag{C 4}$$

where

$$\alpha_L \equiv -\tfrac{1}{2}(\sqrt{(1+4\Gamma_L)}-1) = -\beta_L,$$

and that

$$\Gamma_L = \beta_L^2+\beta_L. \tag{C 5}$$

Clearly, a pseudostationary solution for Γ_L is not obtained; the value of Γ_L depends on the value of the initial non-dimensional concentration gradient β_L. Only positive values of β_L are acceptable.

Although it is possible to obtain solutions to (C 1), (C 2) and (C 3) for arbitrary values of ϵ, it is appropriate to consider only cases in which $\epsilon \ll 1$, because this is a premise on which the derivation of these equations was based. When $\epsilon \ll 1$, the quantities f and Γ in (C 1) will be expressible in the forms $f = f_L+\epsilon f_1$ and $\Gamma = \Gamma_L+\epsilon\Gamma_1$, where f_1/f_L and Γ_1/Γ_L are of order unity. We shall also permit β to differ from β_L to order ϵ, writing $\beta = \beta_L+\epsilon\beta_1$. The first-order expansions of (C 1), (C 2) and (C 3) then become

$$\frac{df_1}{d\eta}-\frac{d^2f_1}{d\eta^2}+\Gamma_L f_1+\Gamma_1 f_L = -\frac{dg}{d\eta}, \tag{C 6}$$

$$\frac{dg}{d\eta}-\frac{d^2g}{d\eta^2}+\Gamma_L g = -\frac{df_L}{d\eta}, \tag{C 7}$$

$$\left.\begin{aligned} f_1 = 0, \quad df_1/d\eta = -\beta_1 \quad \text{and} \quad g = g_0 \quad &\text{at} \quad \eta = 0, \\ f_1 = g = 0 \qquad\qquad\qquad\qquad &\text{at} \quad \eta = \infty. \end{aligned}\right\} \tag{C 8}$$

By substituting (C 4) and (C 5) into (C 7), it is readily found that the desired solution for g is

$$g = \left(g_0 + \frac{\beta_L}{1+2\beta_L}\eta\right) e^{-\beta_L \eta}. \tag{C 9}$$

Substitution of this result and of (C 4) and (C 5) into (C 6) leads eventually to the result that in order to satisfy (C 6) and (C 8), one must have both

$$f_1 = \left[\frac{\beta_L^2 \eta^2/2}{(1+2\beta_L)^2} - \beta_1 \eta\right] e^{-\beta_L \eta} \tag{C 10}$$

and

$$\Gamma_1 = -\frac{\beta_L(1+\beta_L)}{(1+2\beta_L)^2} + \beta_L g_0 + \beta_1(1+2\beta_L). \tag{C 11}$$

Equation (C 11) provides the desired solution for the effect of turbulence on the flame speed.

To observe the meaning of (C 11), it is convenient to represent the result as the fractional change in flame speed produced by turbulence. In view of the definitions of Γ and of Γ_1 this fractional change is expressible as

$$(\bar{v}_T - v_L)/v_L = -(\Gamma - \Gamma_L)/2\Gamma_L = -\epsilon\Gamma_1/2\Gamma_L$$

when it is small. Equations (C 5) and (C 11) therefore imply that

$$(\bar{v}_T - v_L)/v_L = \tfrac{1}{2}\epsilon(1+2\beta_L)^{-2}\left[1 - \frac{g_0(1+2\beta_L)^2}{(1+\beta_L)} - \frac{\beta_1(1+2\beta_L)^3}{\beta_L(1+\beta_L)}\right]. \tag{C 12}$$

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